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PROPERTY C AND CLOSED MAPS

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A standard theorem from dimension theory states that a closed $(m+1)$ to 1 map defined on a finite dimensional space can raise dimension by at most m. Dimension raising maps on countable dimensional spaces and on weakly infinite dimensional spaces have been investigated by A.V. Arhangelskii, A.I. Vainstein and E.G. Sklyarenko. A typical theorem is that a closed map on such spaces raises dimension only if some point has an uncountable number of preimages. A class of infinite dimensional spaces closely related to the two types mentioned above is the class of C spaces. R. Pol's example in 1980 and work of F.D. Ancel have generated renewed interest in C spaces. We prove results about dimension raising closed maps defined on C spaces that are analogous to the results mentioned above.

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Property C	dimension raising map
countable dimensional	weakly infinite dimensional

1. Introduction

Property C, introduced by Haver in [5], is a covering property that all countable dimensional spaces have. This property was further investigated in [1], [6] and [7]. R. Pol's example in 1980 showed that Property C is not equivalent to countable dimensionality [8]. Ancel in [2] shows that a cell-like map defined on a finite dimensional ANR does not raise dimension if the image has Property C. Thus, any CE dimension raising map must have a non C space for its image. Strongly infinite dimensional spaces are examples of such non C spaces.

C spaces are also related to weakly infinite dimensional spaces. In [1], it is shown that every C space is weakly infinite dimensional. It is unknown whether Property C is equivalent to weak infinite dimensionality. In Section 3, we give the known implications for various types of weakly infinite dimensional spaces and discuss known results about 'dimension raising' maps on such spaces. In Section 4, we prove results about dimension raising closed maps on C spaces analogous to the known results on other types of weakly infinite dimensional spaces.

2. Definitions and preliminaries

All spaces are assumed to be separable metric. A space X has *Property C* or is a *C space* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ satisfying the following conditions:

(1) Each $\mathcal{V}_i$ is a pairwise disjoint collection of open sets,

(2) Each V in $\mathcal{V}_i$ is contained in some U in $\mathcal{U}_i$, and

(3) $\bigcup \mathcal{V}_i$ is a cover of X.

Such a sequence $(\mathcal{V}_i)$ is called a *C refinement* of the sequence of covers $(\mathcal{U}_i)$.

A space X is *countable dimensional* if it is a countable union of 0-dimensional spaces. A space X is *strongly infinite dimensional* if there exists a collection $\{(A_i, B_i) \mid 1 \leq i < \infty\}$ of pairs of disjoint closed subsets of X so that if for each i, S_i separates A_i from B_i in X, then $\bigcap S_i \neq \emptyset$. A space X is *weakly infinite dimensional* if it is not strongly infinite dimensional. For more information consult [10].

A topological property P is *hereditary* if every subspace of X has the property P. The property P is *preserved by countable unions* if whenever a space X is a countable union of subspaces each having the property P, then X has the property P. The dimension of a space X, $\dim(X)$, is covering dimension as discussed in [4]. The cardinality of the continuum is represented by c.

3. Dimension raising closed maps on weakly infinite dimensional spaces

The basic setting that we are interested in is the following.

Let P be a topological property and let f be a closed map from a space X with P onto a space Y without P. Let $Y^* \equiv \{y \in Y \mid f^{-1}(y) \text{ has cardinality} \geq c\}$. What can be said about Y^*?

In this setting, if P is a property that represents a type of dimension, a map f as above is called a *dimension raising map.*

The properties of dimension raising maps on countable dimensional spaces and on weakly infinite dimensional spaces have been extensively investigated. The following results are known with respect to the setting indicated above.

(1) If P is the property of countable dimensionality and if X is separable metric, then Y^* is nonempty [11] and in fact is not countable dimensional [3].

(2) If P is the property of weak infinite dimensionality and if X is compact metric, then Y^* is nonempty [12] and in fact is infinite dimensional [13].

Note: Since weak infinite dimensionality is hereditary to closed subspaces and is preserved by countable unions, result (2) can be strengthened to apply to σ compact metric spaces.

The following diagram indicates the relationships among various types of weakly infinite dimensional spaces.

$$\begin{tikzcd}
\text{Countable dimensional} \arrow[d, "1"] & \\
\text{Hereditarily Property } C \arrow[r, shift left, "4"] \arrow[d, "2"] & \text{Property } C \arrow[l, shift left, "/" marking] \arrow[d, "3"] \\
\text{Hereditarily weakly infinite dimensional} \arrow[r, shift left, "5"] & \text{Weakly infinite dimensional} \arrow[l, shift left, "/" marking]
\end{tikzcd}$$

Implication 1 follows from [5], implications 2 and 3 from [1] and implications 4 and 5 follow from the definitions. R. Pol's example shows that none of the conditions on the right half of the diagram imply any of the conditions on the left half [8]. The implications not indicated remain open.

All of the properties on the left half of the diagram are hereditary and are preserved by countable unions. The properties on the right side of the diagram are preserved by countable unions, but are only hereditary to closed subsets. See [1], [13] and [8] for additional details.

The most general theorem for spaces on the left side of the above diagram is the following theorem of Vainstein.

Theorem [13]. *Let P be a topological property that is hereditary and is preserved by countable unions. Let f be a closed map from a separable metric space with P onto a metrizable space Y without P. Then Y^* does not have P.*

As a corollary to this, one sees that if P is any property on the left side of the above diagram, then Y^* is infinite dimensional and one obtains information about the type of infinite dimensionality that Y^* possesses.

4. Main results

The special case of the following lemma where P is weak infinite dimensionality and where X is compact was presented in [13]. This lemma will be used to get the main result we are interested in, Theorem 2.

Lemma 1. *Let D be a class of separable metric spaces so that if $X \in D$ and if A is a closed subspace of X, then $A \in D$. Let P be a topological property satisfying the following conditions:*

(i) *P is hereditary to closed subsets,*

(ii) *P is preserved by countable unions,*

(iii) *All* 0*-dimensional spaces have P, and*

(iv) *If f is any closed map from a space X in D with P onto a space Y without P, then Y^* is nonempty.*

Then if f is any map as in (iv), *Y^* is infinite dimensional.*

Proof. There must be a point y in Y and a neighborhood U of y so that for each neighborhood V of y with $V \subset U$, the boundary of V, $\mathrm{Bd}(V)$, does not have P. Otherwise, Y would have a countable base $\{U_i\}$ so that each $\mathrm{Bd}(U_i)$ had P. This together with conditions (ii) and (iii) above would imply that Y has P. Fix such a point y and a neighborhood U.

The proof will inductively show that $\dim(Y^*) \geqslant n$ for each nonnegative integer n. Condition (iv) implies $\dim(Y^*) \geqslant 0$. Assume inductively that $\dim(Y^*) \geqslant n-1$ whenever f is a map as in condition (iv). For any neighborhood V of y with $V \subset U$, let $A \equiv f^{-1}(\mathrm{Bd}(V))$. Then since A is closed in X, the map $f|A: A \to \mathrm{Bd}(V)$ satisfies condition (iv). The inductive assumption now implies that $\dim(Y^* \cap \mathrm{Bd}(V)) \geqslant n-1$. Let W be the space $Y^* \cup \{y\}$. Since all sufficiently small neighborhoods of y in W have boundaries of dimension $\geqslant n-1$, it follows that W and therefore Y^* is of dimension $\geqslant n$. $\square$

Theorem 1. *Let f be a closed map from a σ compact separable metric C space X onto a non C space Y. Then Y^* is nonempty.*

Proof. If the theorem is true when X is compact, then it is true when X is σ compact. This follows from the fact that Property C is hereditary to closed subsets and is preserved by countable unions. So we assume X is compact. As in [12], it suffices to show that there is a closed non C subspace Y_1 of Y and closed disjoint subspaces X_1 and X_2 of X so that $f(X_1) = f(X_2) = Y_1$. An inductive argument then produces a point $y \in Y$ whose inverse image has cardinality c.

Let $\mathcal{U} = (\mathcal{U}_i)$, $1 \leqslant i < \infty$, be a sequence of open covers of Y that does not have a C refinement. Let $\mathcal{X} = (f^{-1}(\mathcal{U}_i))$, be the corresponding sequence of open covers of X. Since X is a C space and is compact, there exists a C refinement $\mathcal{V} = (\mathcal{V}_i)$, $1 \leqslant i \leqslant N$, of $\mathcal{X}$ where N is a nonnegative integer and each $\mathcal{V}_i = \{V_{i1}, V_{i2}, \ldots, V_{in(i)}\}$. Let

$$M_1 = \bigcap_{j=1}^{n(i)} f(X \setminus V_{1j})$$

and let

$$M_2 = f\left(\bigcap_{j=1}^{n(i)} (X \setminus V_{1j})\right).$$

Then M_1 and M_2 are closed subsets of Y with $M_2 \subset M_1$. In addition, M_1 is not a C space because the sequence of covers $(\mathcal{U}_i)$, $2 \leqslant i < \infty$, restricted to M_1, does not have a C refinement.

If $N = 1$, each V_{1j} is open and closed in X and $M_1 = \bigcup_{i \neq j} M_{ij}$ where $M_{ij} = f(V_{1i}) \cap f(V_{1j})$. At least one M_{ij} must be a non C subspace of Y. For this fixed i and j, let

$Y_1 = M_{ij}$, let $X_1 = f^{-1}(Y_1) \cap V_{1i}$ and let $X_2 = f^{-1}(Y_1) \cap V_{1j}$. Then these are the required subspaces.

The proof will proceed by induction on N. Assume the following statement is true when $K < N$. We will then show that it is true when $K = N$.

Let f be a map from a compact C space X onto a non C space Y. Let $(\mathcal{U}_i)$, $1 \leqslant i < \infty$, be a sequence of open covers of Y with no C refinement. If there exists a C refinement $(\mathcal{V}_i)$, $1 \leqslant i \leqslant K$, of the sequence $(f^{-1}(\mathcal{U}_i))$, then there are closed subsets X_1 and X_2 of X and a closed non C subspace Y_1 of Y so that $f(X_1) = f(X_2) = Y_1$.

To complete the induction in the case $K = N$, there are two cases to consider. Let $\mathcal{S}$ be a sequence of covers $(\mathcal{U}_i), 2 \leqslant i < \infty$, restricted to M_2.

Case 1. The sequence $\mathcal{S}$ does not have a C refinement.

Let $Z = X \setminus \bigcup \mathcal{V}_1$. Then $f(Z) = M_2$. The sequence $\mathcal{S}$ is such that the sequence $f^{-1}(\mathcal{S})$ has a C refinement $(\mathcal{V}_i), 2 \leqslant i \leqslant N$. The inductive assumption now produces a closed non C subspace Y_1 of M_2 and closed disjoint subspaces X_1 and X_2 of Z so that $f(X_1) = f(X_2) = Y_1$.

Case 2. The sequence $\mathcal{S}$ does have a C refinement.

Let $(\mathcal{W}_i)$, $2 \leqslant i \leqslant J$, be a C refinement of $\mathcal{S}$. Let $\mathcal{W} = \bigcup \mathcal{W}_i$ and let $W = \bigcup \mathcal{W}$. Then $M_3 = M_1 \setminus W$ is closed and the sequence of open covers $(\mathcal{U}_i)$, $J+1 \leqslant i < \infty$, restricted to M_3, has no C refinement. Note that for each point $y \in M_3, f^{-1}(y)$ is contained in $\bigcup \mathcal{V}_1$, but is contained in no single V_{1j}. For each j, $1 \leqslant j \leqslant n(1)$, let $D_j = V_{1j} \cap f^{-1}(M_3)$. Then each D_j is compact and $M_3 = \bigcup_{i \neq j} E_{ij}$ where $E_{ij} = f(D_i) \cap f(D_j)$. As in the case $N = 1$, at least one E_{ij} must be a non C subspace of Y. For this fixed i and j let $Y_1 = E_{ij}$, let $X_1 = f^{-1}(Y_1) \cap D_i$ and let $X_2 = f^{-1}(Y_1) \cap D_j$. Then these are the required subspaces. □

The next Theorem follows directly from Lemma 1 and Theorem 1.

Theorem 2. *Let f be a closed map from a σ compact separable metric C space X onto a non C space Y. Then Y^* is infinite dimensional.*

Note: Rohm has shown that weak infinite dimensionality is equivalent to a covering property similar to Property C [9]. The difference is that binary open covers rather than arbitrary open covers are used. Using Rohm's characterization, the proof of Theorem 1 gives a proof of the corresponding theorem about weakly infinite dimensional spaces presented in [12]. The proof in [12] uses the definition of weak infinite dimensionality given in Section 2.

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